Homework 10: Solutions

7.5.8: If S is the triangle with vertices $(1,0,0),(0,2,0),(0,1,1)$, evaluate:

$$\iint_S xyz\, dS.$$

Since the triangle can be thought of as a graph of a function on a triangle in the xy-plane, we will solve this problem like in example 4 of section 7.5.

Let's first find the unit normal vector $\mathbf{n}$ to the triangle S. Taking $(0,2,0)$ as the base, we find two vectors in the plane as $(0,-1,1)$ and $(1,-2,0)$. Thus a normal vector to the triangle S is given by

$$(0,-1,1)\times(1,-2,0)=(2,1,1).$$

And the unit normal vector we're looking for is: $\frac{1}{\sqrt{6}}(2,1,1)$.

This gives us two things: first, the number $\mathbf{n}\cdot\mathbf{k}$ that we need to divide by is $1/\sqrt{6}$ and second, the equation of the plane that S is a part of is

$$2x+y+z=a,$$

for some a, which we find by plugging in $(1,0,0)$ into the equation to be $a=2$. Thus, in particular, $z=2-2x-y$.

Moreover, the projection of S to the xy-plane is the triangle D with vertices $(1,0),(0,1)$ and $(0,2)$. So we evaluate:

$$
\begin{aligned}
\iint_S xyz\,dS &= \iint_D \frac{xyz}{\mathbf{n}\cdot\mathbf{k}}\,dA \\
&= \sqrt{6}\int_0^1 \int_{1-x}^{2(1-x)} xy(2-2x-y)\,dy\,dx \\
&= \sqrt{6}\int_0^1 \left[xy^2 - x^2y^2 - \frac{1}{3}xy^3\right]_{1-x}^{2(1-x)} dx \\
&= \sqrt{6}\int_0^1 \left[3x(1-x)^2 - 3x^2(1-x)^2 - \frac{7}{3}x(1-x)^2\right] dx \\
&= \sqrt{6}\int_0^1 \left(\frac{2}{3}x - 2x^2 + 2x^3 - \frac{2}{3}x^4\right) dx \\
&= \sqrt{6}\left[\frac{1}{3}x^2 - \frac{2}{3}x^3 + \frac{1}{2}x^4 - \frac{2}{15}x^5\right]_0^1 = \frac{\sqrt{6}}{30}.
\end{aligned}
$$

7.5.10: If S is the boundary of the unit ball, evaluate:

$$\iint_S (x+y+z)\,dS.$$

As many people noticed, this is a function on the sphere with a negative value for every positive value of the same magnitude, and since we're just finding the (signed) volume under this function, it must be 0. [Note that we're not calculating the flux of a vector field through the sphere]. We will still compute it explicitly to show that it is 0.

We parametrize the sphere as usual with spherical coordinates, find $\|\Phi_\phi \times \Phi_\theta\|$ and evaluate:

$$x = \cos\theta\sin\phi,\ y = \sin\theta\sin\phi,\ z = \cos\phi$$

$$\|\Phi_\phi \times \Phi_\theta\| = \sin\phi$$

$$
\begin{aligned}
\iint_S (x+y+z)\,dS &= \int_0^\pi \int_0^{2\pi} (\cos\theta\sin\phi + \sin\theta\sin\phi + \cos\phi)\ \sin\phi\ d\theta\ d\phi \\
&= \int_0^\pi \int_0^{2\pi} (\cos\theta\sin^2\phi + \sin\theta\sin^2\phi + \cos\phi\sin\phi)\ d\theta\ d\phi \\
&= \int_0^\pi \left[\sin\theta\sin^2\phi - \cos\theta\sin^2\phi + \theta\ \cos\phi\sin\phi\right]_0^{2\pi}\ d\phi \\
&= 2\pi \int_0^\pi \cos\phi\sin\phi\ d\phi \\
&= \pi\left[\sin^2\phi\right]_0^\pi = 0.
\end{aligned}
$$

7.6.3: If S is

(a) the upper hemisphere of radius 3, or
(b) the sphere of radius 3,

and $\mathbf{F}$ is the vector field $\mathbf{F} = (x, y, z)$, evaluate:

$$\iint_S \mathbf{F}\cdot d\mathbf{S}.$$

Here we are calculating the flux, and since the vector field is always pointing out of the sphere, so the answer should be positive in both cases. Parametrize the sphere as:

$$\Phi(\phi,\theta) = (3\cos\theta\sin\phi,\ 3\sin\theta\sin\phi,\ 3\cos\phi)$$

$$\Phi_\phi \times \Phi_\theta = (9\cos\theta\sin^2\phi, 9\sin\theta\sin^2\phi, 9\sin\phi\cos\phi)$$

Now compute for the hemisphere S^+:

$$
\begin{aligned}
\iint_{S^+} \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2} (3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi)\cdot\Phi_\phi\times\Phi_\theta\ d\phi\ d\theta \\
&= 27\int_0^{2\pi}\int_0^{\pi/2} (\cos^2\theta\sin^3\phi + \sin^2\theta\sin^3\phi + \cos^2\phi\sin^2\phi)\ d\phi\ d\theta \\
&= 27\int_0^{2\pi}\int_0^{\pi/2} (\sin^3\phi + \cos^2\phi\sin\phi)\ d\phi\ d\theta \\
&= 27\int_0^{2\pi}\int_0^{\pi/2} \sin\phi\ d\phi\ d\theta \\
&= 54\pi\left[-\cos\phi\right]_0^{\pi/2} = 54\pi.
\end{aligned}
$$

Whereas, for the full sphere S we get (similarly):

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= 27 \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta \\ &= 54\pi \left[-\cos\phi \right]_0^{\pi} = 108\pi.\end{aligned}$$

7.6.4: If S is the cylinder $x^2 + y^2 = 4, z \in [0,1]$, and $\mathbf{F}$ is the vector field $\mathbf{F} = (2x, -2y, z^2)$, evaluate:

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Here we parametrize by:

$$\Phi(\theta, z) = (2\cos\theta, 2\sin\theta, z),$$

and find that

$$\begin{aligned}\Phi_\theta &= (-2\sin\theta, 2\cos\theta, 0), \\ \Phi_z &= (0, 0, 1), \\ \Phi_\theta \times \Phi_z &= (2\cos\theta, 2\sin\theta, 0)\end{aligned}$$

We also find that:

$$\mathbf{F} \cdot \Phi_\theta \times \Phi_z = (4\cos\theta, -4\sin\theta, z^2) \cdot (2\cos\theta, 2\sin\theta, 0) = 8(\cos^2\theta - \sin^2\theta)$$

Recall, $\cos^2\theta - \sin^2\theta = \cos(2\theta)$. We thus compute:

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^{1} 8\cos(2\theta) dz \, d\theta \\ &= \int_0^{2\pi} 8\cos(2\theta) \, d\theta \\ &= 4\sin(2\theta)|_0^{2\pi} \, d\theta = 0.\end{aligned}$$

7.6.6: If $T(x, y, z) = x$, find the heat flux through the unit sphere, and interpret your answer.

When the temperature is given by T, the heat flow is given by $-k\nabla T$ for some k, a positive constant. Thus in our case the heat flow is $(-k, 0, 0)$. We compute the flux:

$$\begin{aligned}
\iint_S (-k, 0, 0) \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^{\pi} (-k, 0, 0) \cdot \Phi_\phi \times \Phi_\theta \; d\phi \; d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} (-k, 0, 0) \cdot (\cos\theta \sin^2\phi, \sin\theta \sin^2\phi, \sin\phi\cos\phi) \; d\phi \; d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} -k \cos\theta \sin^2\phi \; d\phi \; d\theta \\
&= -k \int_0^{2\pi} \cos\theta \; d\theta \int_0^{\pi} \sin^2\phi \; d\phi \\
&= 0.
\end{aligned}$$

Since the sphere is closed, and there is no source of heat inside the sphere, there is no net flux through the sphere.

7.6.19: The velocity field of a fluid is $\mathbf{F} = (1, x, z)$ m/s. Compute how many cubic meters per second are crossing the upper unit hemisphere.

With the sphere parametrized as before, we compute:

$$\begin{aligned}
\iint_S (1,x,z)\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2} (1,x,z)\cdot \Phi_\phi \times \Phi_\theta \; d\phi \; d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} (1,\cos\theta\sin\phi,\cos\phi)\cdot(\cos\theta\sin^2\phi,\sin\theta\sin^2\phi,\sin\phi\cos\phi)\; d\phi\; d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \cos\theta\sin^2\phi + \cos\theta\sin\theta\sin^3\phi + \sin\phi\cos^2\phi \; d\phi\; d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} 0 + \frac{1}{2}\cos(2\theta)\sin^3\phi + \sin\phi\cos^2\phi\; d\phi\; d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} 0 + 0 + \sin\phi\cos^2\phi\; d\phi\; d\theta \\
&= -2\pi\frac{\cos^3\phi}{3}\Big|_0^{\pi/2} \\
&= -\frac{2\pi}{3}(0-1) = \frac{2\pi}{3}
\end{aligned}$$

4.4.4: Find the divergence of $\mathbf{V} = (x^2,(x+y)^2,(x+y+z)^2)$

Divergence of $\mathbf{V}$ is given by:

$$\nabla\cdot\mathbf{V} = 2x + 2(x+y) + 2(x+y+z) = 6x+4y+2z.$$

4.4.18: Find the scalar curl of $\mathbf{F} = (y,-x)$

The scalar curl of the vector field $\mathbf{F}(x,y) = (P(x,y),Q(x,y))$ is the number $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. Thus we find the scalar curl of $\mathbf{F}$ is equal to $(-1)-1=-2$.

4.4.18: Suppose $\nabla\cdot\mathbf{F}=0$ and $\nabla\cdot\mathbf{G}=0$. Which of the following necessarily have zero divergence?

(a) $\mathbf{F}+\mathbf{G}$

(b) $\mathbf{F} \times \mathbf{G}$

Note that

$$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G} = 0$$

So $\mathbf{F} + \mathbf{G}$ necessarily has zero divergence.

As for $\mathbf{F} \times \mathbf{G}$, this is not necessary, as can be shown by the simple counterexample:

$$\mathbf{F} = (-y, x, 0)$$

$$\mathbf{G} = (0, 0, 1)$$

$$\mathbf{F} \times \mathbf{G} = (x, y, 0)$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = 2 \neq 0$$